AN ANALYSIS OF STUDENTS' IDEAS ABOUT TRANSFORMATIONS OF FUNCTIONS

Ana Elisa Lage
Arizona State University

Maria Trigueros Gaisman
Instituto Tecnológico Autónomo de México
trigue@itam.mx

This study intends to contribute to better understand students' difficulties with transformations of functions. Students were interviewed while solving problems involving transformations of functions. Results were analyzed using APOS theory (Asiala, et al., 1996). They show that few students can work confidently with these problems because they do not seem to have interiorized the processes involved in transformations, or encapsulated those processes into objects.

The study of families of functions and transformations on them (such as translations, reflections and stretches) has become important in pre-calculus courses at universities. Transformations give students new opportunities to use and reflect on the concept of function, and they can become a useful tool in more advanced topics of mathematics.

Research on students' understanding of transformations of functions is important not only because it is a topic in many pre-calculus courses, but also because it provides an opportunity to analyze students' ideas on functions and variables. It also permits to study students' flexibility with the use of different representations of function. All this information can be used as a guide in the design of teaching materials and class strategies with the purpose of trying to foster students' understanding of both functions and transformations of functions.

Antecedents

Some researchers have worked on the concept of transformations of functions (Baker, et al., 2001a; Baker, et al.; Bloch, 2000; Cuoco, 1994; Eisenberg & Dreyfuss, 1994; Quiroz, 1990; Goldenberg, 1988; Zazkis, 2003). They have found specific difficulties students have when working with a few particular problems where transformations are involved. Research focusing on a wider variety of situations is needed to further analyze how students work with situations involving transformations of functions and how their understanding of this concept relates to their concept of function.

This study focuses on the concept of transformations, using APOS theory (Asiala, et al., 1996) as a theoretical framework. A genetic decomposition for the concept of transformations of functions developed in previous research, was refined to analyze students' work and to find possible causes for their difficulties. (Baker, et al. 2001a).

The purposes were to investigate how students use transformations of functions once they have finished a pre-calculus course, to find out what their main difficulties when using them are, and determining the conditions that identify those tasks where they are able to succeed.

Research Questions

This study intends to respond the following research questions:

- Are students able to indentify when a given function can be described in terms of a basic transformed function?

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

- What are the specific difficulties that students face when they work with transformations of functions? Which of those difficulties are related with the use of different representational contexts?
- What can be said about students' understanding of the concept of function from their work with transformations of functions?

Theoretical Framework

The theoretical framework used for this study is APOS (action, process, object, schema) theory (Asiala et al., 1996). The genetic decomposition of the concept of transformation of functions used was a refinement of that developed by Baker and her collaborators (Baker, et al., 2001a) with the purpose of analyzing the data obtained in this study. The genetic decomposition used for the analysis of the data follows:

Students who act at an action conception of transformation of functions can perform operations on functions and variables step by step, and these operations can be applied either in the analytical or graphical representation context; rely on memorized facts or external signs, as for example the exponents in the expressions or the apparent form of the graph; recognize differences between a function and its transformations only in terms of the syntax of the rule that defines the function, and recognize similarities between a function and its transformations, or between transformations, only in terms of some global property of the graph. When these actions are repeated on the analytical or graphical representation of a function, and students reflect upon them, they interiorize the actions into **a process**.

Students who act at a process level are able to describe changes in the basic functions as a consequence of the application of the transformation without the need to perform each step of the transformation or move the graph of a function step by step. They are able to look at the graph of the transformed function and describe the changes that result from the transformation. These students are also able to reverse the process to identify the function on which a set of transformations was applied. Students at this level show, however, difficulties in coordinating the information obtained from different representational contexts, and in flexibly translating information from one representational context to another.

When students reflect on all of these processes, and are able to think of them as a whole, in any representational context, working flexibly in different representational contexts as well, it is considered that they have encapsulated the process of applying a transformation to any function into an object. It is considered that students have an object conception of transformation, if they are able to apply actions on transformed functions and coordinate their properties in terms of possible changes in the original function, At this level, students are able to de-encapsulate any transformed function object into the process involved in its construction, and they are able to identify the basic function on which it is based and compare different transformed functions in terms of their properties in any representational context.

Schema conceptions are formed by the interconnection of several actions, processes, objects and other previously constructed schema, and the relationship between them. One possible example for a transformation of functions schema would include the schema for function, the transformation of functions object, and actions and processes on transformed functions to determine their properties or to classify them, for example into rigid or non-rigid transformations. Since students in this project had only taken one course related to the notion of transformation, it was decided by the researchers to use solely the concepts of action, process, and object from APOS in the analysis of students' responses and work.

Methodology

The genetic decomposition outlined above was used to design and analyze the actions processes and objects the researchers considered were required in the responses to the questions from an instrument designed to diagnose students' understanding of the concept of transformations of functions. This instrument was used as a questionnaire that was solved by 158 students registered in pre-calculus courses and calculus courses at a small private university in México City. The questionnaire included eleven questions where students had to work with transformations of functions using both, analytical and geometrical representations. Students' work was then analyzed comparing the constructions they showed in their answers, with those predicted by the genetic decomposition. From the analysis of students' answers to the questionnaire, 16 students were selected to be interviewed. Half of them had finished a pre-calculus course based on transformations of functions and the rest had finished a calculus course. The selection was based on students' responses to the questionnaire and the overall tendency found in their responses, in order to end up with an even number of students classified at each level according to APOS theory.

The same instrument was then used during the semi structured interviews with the purpose of conducting a more in depth investigation of their understanding of transformations of functions and to uncover the strategies they used while working with different issues related to this concept. The interview included an extra question which could only be solved in a geometrical context, and where students were expected to apply their knowledge about transformations of functions to solve an inequality. Interviews were audio taped, transcribed and analyzed qualitatively, using the criteria derived from the genetic decomposition of the concept. The analysis was done separately by two researchers. Discrepancies in the analysis were negotiated after another round of analysis by the same two researchers.

The questions of the interview were classified in four groups, according to their main global purpose. These four groups were: Identification of transformations, graphing transformed functions, using transformations of functions to perform some other actions or processes, and finding the domain and range of transformed functions. Students' difficulties were associated with these four groups in order to determine those characteristics of problems that appeared to make them troublesome for students. On the other hand, students were grouped according to their difficulties and strategies of solution, and to the actions, processes and objects they showed a tendency to use while working on all the questions. It is important to note that these terms are only used here for classification purposes, since it is acknowledged that the same student can perform at an action level in some question, and at a process or object level on others, depending on both, his or her cognitive constructions and the requirements of the question.

Results

The classification of students resulted in 7 students working at an action level, 5 students working at a process level and 4 students at an object level.

The data obtained for the group of questions related to the **identification of transformations** show that when a problem involves a function which can be the result of the application of a set of transformations on a basic function, students are generally able to identify some of the transformations that have been applied. Students who had difficulties with this type of tasks found troublesome to associate a function represented graphically with its corresponding analytical representation. They showed a tendency to use memorized facts or to make a table of data in order to succeed in these tasks. For example to a question showing the graph of a parabola which asked, a) find the values of a and b in $f(x)=(x-b)^2 + a$ and, b) what would happen for different values of a and b, a student at an action level responded "*I know a moves it up and down… the other, when it is inside… I cannot remember…I would have to make a table with values and see…*"and he proceeded to make a table, but had difficulties to infer the values from it. A smaller group of students showed a tendency to generalize the conservation of distance property, which is valid for rigid transformations, to non-rigid transformations, demonstrating that they apply actions or processes to the graph of the function as if it was a rigid object, with no reflection on what the result of those actions would be for each particular point on the domain of the function. The following is a response given by a student working at an action level: *"…I know that multiplying by 3 makes the graph narrower, it moves this way, closer to the Y axis…so* $f(x) = 3\sqrt{x-2}$ *the graph will be like…more stretched than the original…"*. In questions related to rigid transformations, students had more difficulties recognizing a horizontal translation than a vertical translation. This seems to be due to the fact that students memorize the rules for transformations and their corresponding effect on the function. When remembering this information students often associate the incorrect direction to the translation. The most difficult question in this group was related to the identification of similarities and differences between functions where the same set of transformations had been applied. This question was taken from Baker et al. (2001a). A student working at a process level showed that she had interiorized the result of applying different transformations to a well known function, but she did not recognize the similarities in terms of the transformations applied to both curves: *"I can see this is a parabola that has been moved 3 units up and 5 to the right, and it is stretched by a factor of 2…, the differences,… the other is a hyperbola not a parabola and has asymptotes …they don't touch."*. Students who worked at an action level on this group of tasks based their explanations on the differences they perceived in the algebraic formula for the function. Students who worked at a process level were able to work with functions in different representational contexts, and referred, in their explanations, to the process involved in the transformation. Students working at an object level were able to identify families of functions in any representational context, could relate graphs with their analytic representation and vice versa, could explain which characteristics of a function are conserved, and which are not, when a set of transformations is applied, and identified similarities and differences when comparing transformed functions.

Questions related to **graphing transformations** included questions where students had to graph a given function and questions where transformations had to be applied to a function in a graphical context. Students were, in general, able to apply or graph transformed functions only when one transformation had been applied, or when the basic function was very familiar to them. But, again, as in the case of identification of transformations, they had more difficulties when the transformation involved was a horizontal translation. When a stretch transformation was applied, many students referred to the graph of the function as being narrower or wider, depending of the factor of the stretch, but they were not able to explain changes to the original function beyond those they remembered from memorized rules. These students worked at an action level and were not able to realize that the rate of change of the function changed due to the transformation applied. They seemed to consider the curve as a "wire" that can be bended without changing the relative distance between its different points. A typical response given by some students working at an action or process level when asked to graph and describe the function f(x)= [2/(x-3)] – 5 was *"…as it is multiplied by 2, it is narrower. I: What do you mean by narrower? S: its graph is not as spread out as without the 2. I: is that true for any point in the domain of the function? S:…mmm, yes, it always does the same"*. Most students found it difficult to reflect the graph of a function over the Y axis, and even more when the transformation applied contained an absolute value, as was the case for a function given in graphical representation, where it was needed to find *f(-x)*. Even students working at a process level, for example, interpreted the transformation as a reflection over the X axis, instead of reflecting over the Y axis: "*I: how do you know it goes that way? S: because this minus 1 means that all the points have to be moved to the other side, down here...it is minus x"*. In this last question, the "form" of the curve changes under the transformation, but only students who showed to be working at an object level could explain the result of the transformation. For students working at an action level, applying transformations to functions presented in a graphical representation was almost impossible. They showed strong difficulties with the concept of function itself. They tried to find a "formula" for the curves and then tried to plot that function point by point. They rarely referred to transformations of functions; they did so only when the examples were familiar to them, and they had probably memorized the effects of each transformation. Students working at a process level could graph transformed functions if the transformations that were applied to the original function were rigid. They struggled with other transformations, but consistently explained using transformations the changes of the basic function. Students working at an object level were able to respond correctly and to explain most of the questions in this group. They could graph the transformed functions explaining which transformations could be used, and they were able to recognize the function on which the transformations were applied. They were also able to identify properties of the families of transformed functions.

Questions related to the **use of transformations** of functions were those where transformations of functions are a basic tool to solve a particular problem. For example, students had to relate a graph from an unfamiliar function to its analytical representation they also had to explain how to transform one function into another when both graphical representations were provided, and finally, they had to solve a difficult inequality using transformations. Those students who were found to work at an action level could not solve this group of questions. Students working at a process level demonstrated they had interiorized the actions involved in transforming a function but presented difficulties when working with trigonometric functions or with functions they could not recognize. Students working at an object level could solve these questions even though they struggled and had to reconsider their work in several occasions. On the following excerpts we present examples of answers provided by three different students each working at a different level. These examples show the different approaches they used when they were asked to solve $\frac{1}{x-1}+4\le -\frac{1}{2}(x-2)^2+6$.

Carlos, an action level student said "*I have to solve first the squared part, and develop the expressions to leave x on one side and numbers at the other…*" he tried unsuccessfully to achieve this, then the interviewer suggested "*Can you use graphs of both functions to help you solve the question? S: Well…I would need to…make a table here…but, not really, I don't think I can graph them, this is a parabola, it is squared here, but I don't know… I am not too good at graphing difficult functions…*"

Tania, a process level student, said "*…The way I would solve that is by finding the graphs of the functions…I first thought to use algebra, but then I realized it is easier to graph… the first part, the function is a hyperbola the other a parabola….they would be like that… then…I need to…find when the first is less or equal than the second… and this happens here, from about 0,5 to infinity*". This student could use the transformations to graph both functions but her drawing was not completely correct because she considered that multiplying the parabola by ½ made it narrower, and then did not interpret correctly the intervals. Lucía, a student classified at an object level, replied: "*I will try to do it graphically…graph the function 1/x-1 + 4, and also –1/2(x-2)^2+ 6…two functions… the first is not defined at x=1, it is moved 4 units up, it is like 1/x, a hyperbola, it is also moved 1 to the right… the other is squared, it is a parabola,… it opens down, its vertex is at (2,6), it is wide because of this ½, it grows more slowly than x^2 …now looking at this graph I want f(x) to be less than or equal to g(x)…so…the solution is x from more or less –1 to 1 union from 1.5 to 4 more or less…*". This student was able to use transformations in the solution of a new problem. She was able to identify the type of transformations, graph the transformed functions, and interpret the result in terms of the situation she was trying to solve.

Students' strategies to transform a function and to explain how it could be transformed reflected the same mentioned difficulties. We can conclude from their work that students worked with ease only when vertical translations were applied. When the information was provided in a graphical context, students showed difficulties to identify each transformation, or to predict its effect. Finally, students could use transformations with more ease when they were already familiar with the basic function. The last group of questions involved those where students needed to determine some properties of a family of functions, such as **domain and range**, or simply predict changes in the domain and range when a function goes through a set of transformations. Students working at an action level could determine some domains based on memorized facts, but were in general unable to determine the range of functions. They could only determine domain and range with ease for graphical representations of frequently used functions. Students working at a process level could find domains of almost all the functions that were presented to them. However, they had more difficulties when they had to find the range of those same functions. They often relied on drawing the graph of the function and, as we have already mentioned, they showed some problems to draw them correctly. Students working at an object level were able to determine domain and range of all the functions.

Discussion

Results of this study are consistent and also complement those found in previous research. They show that students' difficulties with the concept of transformation of functions are strongly related to their understanding of the concept of function. It was found that students who were classified at an action level show a weak understanding of the concept of function even though, in some cases, they have already taken a Calculus course. In particular, these students showed conceptual problems when discussing the graphical representation of functions and when looking for their domain and range. Although transformations of functions can be used in the solution of many problems, some students in this sample were not able to use them even in the case of problems that were similar to those studied in the pre-calculus course they had already taken. This situation is probably due to their point by point strategies to graph functions and their reliance on memorized facts. These students' knowledge about functions was not enough to interpret and work with transformations. Only a few of the students of this sample were capable to flexibly use those functions that were presented in a graphical context. Students showed a little more fluency when the function was presented in an analytical representation, that is, when it was presented in terms of a formula for the function; but, even in these cases, students showed lots of difficulties when they were asked about the properties of functions and to predict the properties of functions that change when a transformation is applied.

Even though we would expect that students who had taken a Calculus course would have a better understanding of the concept of transformation of functions, this was not found to be true. There were almost the same number of pre-calculus and calculus students working at the levels of action and process. However, it was found that all the students but one at the object level had already finished a Calculus course.

Transformations of functions can be classified according to the difficulty they present to students. Rigid transformations were easier for students. Students' solution strategies when applying transformations focused on what happened to the function as a whole and not on what happened to each of the points in the domain of the function. Dynamical transformations, that is, transformations where functions are "deformed" presented more problems to students. Their work demonstrated that they had not interiorized the effects of transformations on functions when it was needed to think in terms of co-variation of the dependent and independent variables of the function. Students had troubles when they had to identify which transformation had been applied to a particular basic function. When a transformation was given, they had problems finding its properties. All these difficulties were more apparent when the representation used in the question was graphical.

The results of this study demonstrate that when teaching transformations it is important to consider a wider variety of functions and to explicitly demonstrate the result of applying a transformation both at the level of what happens to the function in general, and what happens to different points in its domain. Many research studies have stressed the need to attain flexibility with the use of different representations in order to understand a variety of concepts. Results of this study show that this is also the case when teaching transformations of functions.

References

Asiala, M., Brown, A., DeVries, D.J., Dubinsky, E., Mathews, D., and Thomas, K. (1996) *A framework for research and curriculum development in undergraduate mathematics education*, Research in Collegiate Mathematics Education, II, 1 - 32.

Baker, B. Hemenway, C. & Trigueros, M. (2001a) On transformations of functions *Proceedings of the XXIII Annual of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Vol. 1. Robert Speiser, Carolyn A. Mahler and Charles N. Walters (Eds.). ERIC Clearinghouse for Science, Mathematics and Environmental Education. Columbus, Ohio. pp. 91-98.

Baker, B., Hemenway, C and Trigueros, M.: (2001b) On Transformations of Basic Functions, *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra*, Vol.1. Helen Chick, Kaye Stacey, Jill Vincent & John. Vincent (eds.), The University of Melbourne, Australia, pp. 41-47.

Bloch, I. (2000) *Le enseignement de l'analyse à la charnière lycée / université Savoirs, connaissances et conditions relatives à la validation.* Doctoral dissertation, Université Bordeaux I, Bordeaux, France.

Cuoco, A. (1994) *Multiple representations of functions*. In J. Kaput, and E. Dubinsky, (Eds.), Research in Collegiate Mathematics Education, I, 121 – 140.

Eisenberg, T. and Dreyfuss, T. (1994) *Learning to Visualize Functions*. Research on Collegiate Mathematics Education _I, 45-68.

Goldenberg, E. P. (1988) *Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representations of functions.* Journal of Mathematical Behavior, 7, 135-173.

Quiroz, M. (1989) *Instalación de un lenguaje gráfico en estudiantes que inician estudios universitarios. Un enfoque alternativo para la reconstrucción del discurso matemático escolar de precálculo*. Tesis de Maestría, CINVESTAV, México.

Zazkis, R. (2003) Translation of a function: Coping with perceived inconsistency. *Proceedings of the XXVII Joint Meeting of PME and PME-NA,*4, 459- 468, Hawaii, USA.